JOURNAL OF COMPUTATIONAL PHYSICS 12, 334–347 (1973)

An Initial Value Method for the Eigenvalue Problem for Systems of Ordinary Differential Equations[1]

MELVIN R. SCOTT

Applied Mathematics Division, 2642
Sandia Laboratories, Albuquerque, New Mexico 87115

Received August 1, 1972

An initial value method is presented for the calculation of characteristic lengths and characteristic functions of systems of ordinary differential equations. Very general nonseparated boundary conditions can be handled and the equations tend to be quite stable numerically. Numerical examples are presented which demonstrate the efficacy of the method.

1. INTRODUCTION

The method of invariant imbedding has been used extensively to compute the characteristic lengths and eigenvalues of second order problems [10, 12, 14]. Recently the calculation of characteristic lengths, eigenvalues and eigenfunctions for N-th order systems of equations has begun to attract more attention, particularly in the areas of structural mechanics [2, 6] quantum mechanics [4, 7] and nuclear physics [3]. It soon became evident that many of the classical techniques, such as the phase function approach and the implicit finite difference scheme, lose some of their appeal since they are either difficult to generalize to systems or require large blocks of storage. Whereas many techniques do not easily generalize to systems, the invariant imbedding approach has a straightforward generalization to systems. The primary feature of the method discussed here is that the boundary value problems encountered are transformed into initial value problems which tend to be quite stable numerically. Hence, the method is ideally suited for implementation on modern computers. The reader is referred to [13] for an extensive bibliography on the subject of invariant imbedding.

In this paper we shall give a general development of the method for systems of equations. In Section 2 we shall derive the method for N-th order systems with simple separated boundary conditions. Then in Section 3 we shall generalize the

[1] Work supported by the U. S. Atomic Energy Commission.

method to handle problems with very general boundary conditions. A brief discussion of multiple eigenfunctions is presented in Section 4. Two pertinent numerical examples are discussed in Section 6 and in the last section we present our conclusions and suggestions for future study.

2. Characteristic Lengths for N-th Order Systems

Let us consider the system

$$\bar{u}'(z) = A(z)\bar{u}(z) + B(z)\bar{v}(z), \tag{2.1a}$$

$$-\bar{v}'(z) = C(z)\bar{u}(z) + D(z)\bar{v}(z), \tag{2.1b}$$

subject to

$$\bar{u}(0) = 0, \tag{2.2a}$$

$$\bar{u}(x) = 0, \tag{2.2b}$$

where $\bar{u}$ and $\bar{v}$ are n-vectors, the coefficients A, B, C, D are $n \times n$ matrix functions of z and a parameter λ. We assume, for a fixed value of λ, that the coefficients are such that there exist a countable number of values of x such that (2.1–2.2) has a nontrivial solution and that all initial value problems for (2.1) have unique solutions

The Riccati transformation for the above system is given by

$$\bar{u}(z) = R_1(z)\bar{v}(z), \tag{2.3}$$

where R_1 is an $n \times n$ matrix function. A set of more general transformations will be discussed in the next section. Although the above transformation is valid, we simply do not have enough information available to derive the matrix differential equation satisfied by $R_1(z)$. In order to circumvent this problem, we must temporarily consider a more general problem. That is, we consider the system

$$U'(z) = A(z)U(z) + B(z)V(z) \tag{2.4a}$$

$$-V'(z) = C(z)U(z) + D(z)V(z), \tag{2.4b}$$

subject to

$$U(0) = 0, \tag{2.5a}$$

$$V(0) = I, \tag{2.5b}$$

where U and V are $n \times n$ matrices, I is the $n \times n$ identity matrix and A, B, C, D are as given in (2.1). In terms of the matrix functions U and V, the transformation in (2.3) becomes

$$U(z) = R_1(z)V(z), \tag{2.6}$$

where $R_1(z)$ is the same as in (2.3). The relationships between $\bar{u}$, $\bar{v}$ and U, V are given by

$$\bar{u}(z) = U(z)\bar{v}(0), \tag{2.7a}$$

$$\bar{v}(z) = V(z)\bar{v}(0). \tag{2.7b}$$

Now we can proceed to derive the matrix differential equation for R_1. Since we are dealing with matrices, the order of operation must be carefully observed. If we differentiate in (2.6), we get

$$U'(z) = R_1'(z)V(z) + R_1(z)V'(z). \tag{2.8}$$

Substituting (2.4a) into (2.8), using (2.6) and simplifying, we get

$$\{R_1'(z) - B(z) - A(z)R_1(z) - R_1(z)D(z) - R_1(z)C(z)R_1(z)\}V(z) = 0. \tag{2.9}$$

Since $V(z)$ is nonsingular, at least in some neighborhood of the origin, it follows that the terms in braces must be zero. (It is now clear why we had to consider the more general matrix equations (2.4–2.5)). Thus we have

$$R_1'(z) = B(z) + A(z)R_1(z) + R_1(z)D(z) + R_1(z)C(z)R_1(z). \tag{2.10}$$

The initial condition for (2.10) is

$$R_1(0) = 0, \tag{2.11}$$

which follows by evaluating (2.6) at $z = 0$.

In order to find the characteristic lengths of the problem (2.1–2.2), we evaluate (2.3) at $z = x$ and we obtain

$$\bar{u}(x) = 0 = R_1(x)\bar{v}(x). \tag{2.12}$$

For a nontrivial solution of (2.1–2.2) to exist, at least one component of $\bar{v}(x)$ must be nonzero. Hence, it follows that at $z = x$

$$\det [R_1(x)] = 0. \tag{2.13}$$

Also, by a reverse argument, it follows that the determinant of $R_1(z)$ is zero only at the characteristic lengths.

We cannot, however, simply integrate the matrix differential equation for $R_1(z)$ until the determinant vanishes. As in the scalar case [10, 14], the Riccati equation has at least one singularity between points where the determinant is zero. It may have more than one due to the possible presence of several linearly independent solutions of (2.1–2.2) for certain values of the parameter λ. Hence, we shall temporarily assume that, for each value of λ, there is only one characteristic function of (2.1–2.2) and defer the discussion of multiple eigenfunctions to Section 4.

In order to avoid the points where $\det[R_1(x)]$ is infinite, we define the inverse transformation by

$$\bar{v}(z) = S_1(z)\bar{u}(z), \tag{2.14}$$

where $S_1(z)$ is an $n \times n$ matrix. In order to derive the differential equation for the $S_1(z)$ function, we must again use the techniques discussed earlier. The equation is

$$-S_1'(z) = C(z) + S_1(z)A(z) + D(z)S_1(z) + S_1(z)B(z)S_1(z), \tag{2.15}$$

On any common interval of definition it is clear that the matrix functions $R_1(z)$ and $S_1(z)$ are related by

$$S_1(z) = R_1^{-1}(z). \tag{2.16}$$

The above relation also serves as the initial condition for $S_1(z)$ after a switch has been performed. The switching is performed any time the absolute value of the determinant of one of the matrix functions exceeds some predetermined, usually large, value.

It is clear that if we evaluate (2.14) at $z = z_1$, where $\bar{v}(z_1) = 0$, that we obtain

$$0 = S_1(z_1)\bar{u}(z_1). \tag{2.17}$$

For this system of equations to have a nontrivial solution, it is necessary and sufficient that

$$\det[S_1(z_1)] = 0 \tag{2.18}$$

Hence, we are also able to simultaneously compute the values of z_i where (2.1) is subject to the boundary conditions

$$\bar{u}(0) = 0, \tag{2.19a}$$

$$\bar{v}(z_i) = 0, \tag{2.19b}$$

has a nontrivial solution.

3. More General Boundary Conditions

If the boundary conditions are complicated, then a set of more general transformations must be introduced. Suppose we wish to find the characteristic lengths for (2.1) subject to the boundary conditions

$$\alpha_1\bar{u}(0) + \beta_1\bar{v}(0) + \gamma_1\bar{u}(x) + \delta_1\bar{v}(x) = 0, \tag{3.1a}$$

$$\alpha_2\bar{u}(0) + \beta_2\bar{v}(0) + \gamma_2\bar{u}(x) + \delta_2\bar{v}(x) = 0, \tag{3.1b}$$

where α_i, β_i, γ_i, δ_i $(i = 1, 2)$ are $n \times n$ matrices.

Since $\bar{u}(0)$ may be nonzero and, in fact, unknown, we must modify our Riccati transformation of (2.3). The new transformation, which we shall refer to as the *generalized Riccati transformation*, is given by

$$\bar{u}(z) = R_1(z)\bar{v}(z) + R_2(z)\bar{u}(0). \tag{3.2a}$$

In addition, we introduce the *recovery transformation*

$$\bar{v}(0) = Q_1(z)\bar{v}(z) + Q_2(z)\bar{u}(0). \tag{3.2b}$$

where R_1, Q_1 are as before and R_2, Q_2 are $n \times n$ matrices which account for the fact that $\bar{u}(0)$ will, in general, not be zero. The differential equations for R_1, R_2, Q_1, and Q_2 are derived in a similar fashion to those derived earlier in Section 2 and they are

$$R_1'(z) = B(z) + A(z)R_1(z) + R_1(z)D(z) + R_1(z)C(z)R_1(z), \tag{3.3a}$$

$$R_2'(z) = [A(z) + R_1(z)C(z)]R_2(z), \tag{3.3b}$$

$$Q_1'(z) = Q_1(z)[C(z)R_1(z) + D_1(z)], \tag{3.3c}$$

$$Q_2'(z) = Q_1(z)C(z)R_2(z), \tag{3.3d}$$

subject to the initial conditions

$$R_1(0) = 0, \tag{3.4a}$$

$$R_2(0) = I, \tag{3.4b}$$

$$Q_1(0) = I, \tag{3.4c}$$

$$Q_2(0) = 0. \tag{3.4d}$$

In order to calculate the characteristic lengths for (3.1) subject to (3.1), we view (3.1) and (3.2) as a system $4n$ equations in the $4n$ unknowns $\bar{u}(0)$, $\bar{v}(0)$, $\bar{u}(x)$ and $\bar{v}(x)$. In matrix form we have

$$\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ R_2(x) & 0 & -I & R_1(x) \\ Q_2(x) & -I & 0 & Q_1(x) \end{bmatrix} \begin{bmatrix} \bar{u}(0) \\ \bar{v}(0) \\ \bar{u}(x) \\ \bar{v}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.5}$$

Thus we seek the values of $z = x$ such that

$$\det(A) = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ R_2(x) & 0 & -I & R_1(x) \\ Q_2(x) & -I & 0 & Q_1(x) \end{vmatrix} = 0. \tag{3.6}$$

Again we must introduce the inverse Riccati and recovery transformations in order to avoid any singularities of the Riccati equation. Thus, at some point z', we write

$$\bar{v}(z) = S_1(z)\bar{u}(z) + S_2(z)\bar{v}(z'), \tag{3.7a}$$

$$\bar{u}(z') = T_1(z)\bar{u}(z) + T_2(z)\bar{v}(z'), \tag{3.7b}$$

where S_1, S_2, T_1, and T_2 are $n \times n$ matrix functions which satisfy the initial value problems

$$-S_1'(z) = C(z) + S_1(z)A(z) + D(z)S_1(z) + S_1(z)B(z)S_1(z), \tag{3.8a}$$

$$-S_2'(z) = [D(z) + S_1(z)B(z)]S_2(z), \tag{3.8b}$$

$$S_1(z') = 0, \tag{3.8c}$$

$$S_2(z') = I, \tag{3.8d}$$

$$-T_1'(z) = T_1(z)[A(z) + B(z)S_1(z)], \tag{3.9a}$$

$$-T_2'(z) = T_1(z)B(z)S_2(z), \tag{3.9b}$$

$$T_1(z') = I, \tag{3.9c}$$

$$T_2(z') = 0. \tag{3.9d}$$

We must rewrite (3.7) in terms of $\bar{u}(0)$ and $\bar{v}(0)$. We do this by evaluating (3.2) at $z = z'$ and solving for $\bar{u}(z')$ and $\bar{v}(z')$. We find

$$\bar{v}(z') = Q_1^{-1}(z')\{\bar{v}(0) - Q_2(z')\,\bar{u}(0)\}, \tag{3.10a}$$

$$\bar{u}(z') = R_1(z')\,Q_1^{-1}(z')\,\bar{v}(0) + \{R_2(z') - R_1(z')\,Q_1^{-1}(z')\,Q_2(z')\}\,\bar{u}(0). \tag{3.10b}$$

Substitution of these values into (3.7) yields

$$\bar{v}(z) = S_1(z)\bar{u}(z) + S_2(z)\{A_1\bar{v}(0) + A_2\bar{u}(0)\}, \tag{3.11a}$$

$$[B_2 - T_2(z)A_2]\,\bar{u}(0) = T_1(z)\bar{u}(z) + [T_2(z)A_1 - B_1]\bar{v}(0), \tag{3.11b}$$

where

$$A_1 = Q_1^{-1}(z'), \qquad A_2 = -Q_1^{-1}(z')\,Q_2(z'),$$

$$B_1 = R_1(z')\,Q_1^{-1}(z'), \qquad B_2 = [R_2(z') - R_1(z')\,Q_1^{-1}(z')\,Q_2(z')].$$

The matrix $Q_1(z)$ is nonsingular since it is the solution of a linear matrix differential equation with nonsingular initial conditions. The boundary conditions (3.1a,b) and Eqs. (3.11a,b) evaluated at $z = x$ form a system of $4n$ equations in $4n$ unknowns, $\bar{u}(0)$, $\bar{v}(0)$, $\bar{u}(x)$ and $\bar{v}(x)$. In matrix form these equations become

$$\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ S_2(x)\,A_2 & S_2(x)\,A_1 & S_1(x) & -I \\ T_2(x)\,A_2 - B_2 & T_2(x)\,A_1 - B_1 & T_1(x) & 0 \end{bmatrix} \begin{bmatrix} \bar{u}(0) \\ \bar{v}(0) \\ \bar{u}(x) \\ \bar{v}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.12}$$

Thus we have that the characteristic lengths are the values of x such that the determinant of the matrix of coefficients in (3.12) is zero. We may summarize our results as a theorem.

THEOREM 3.1. *If the coefficients A, B, C, D are real and such that the solutions of* (2.1*a,b*) *subject to the boundary conditions* (3.1*a,b*) *have a countable number of points x_i for which the solutions are nontrivial, then these points correspond to the points where the determinant of the matrix of coefficients of* 3.5() *or* (3.12) *is zero.*

4. MULTIPLE EIGENFUNCTIONS

In the case of single second order equations there normally exists only one eigenfunction, apart from constant multiples, for each eigenvalue. In fact, the only time that multiple eigenfunctions can occur for single second order equations is when the boundary conditions are nonseparated, such as being periodic [8]. However, for systems the problem is considerably more complicated. There are no simple criteria for determining a priori when an eigenvalue of a general n-th order equation with prescibed boundary conditions will have several linearly independent eigenfunctions. We shall see, however, that the presence of multiple eigenfunctions causes no particular problem in our calculational procedure.

The procedure described in the previous section must be slightly modified since the singularities and zeros of the determinant of the matrix of coefficients will not necessarily separate one another. This simply implies that the switching process should not be performed. The presence of multiple eigenfunctions is easily detected since the multiplicity of the roots at x where the determinant is zero indicates the multiplicity of the linearly independent eigenfunctions.

In order to illustrate the behavior of the determinant of the matrix of coefficients when a degenerate eigenvalue is present we shall consider three examples. Consider the problem

$$y^{iv} = k^4 y, \tag{4.1}$$

$$y(0) = y''(0) = 0 = y(x) = y''(x). \tag{4.2}$$

This particular problem has only simple eigenvalues. For a given value of k, the search for the values of x such that (4.1–4.2) has a nontrivial solution amounts to finding a value of x such that

$$0 = | A | = | R_1 | = (1/k^2) \tan kx \tanh kx. \quad (4.3)$$

In Fig. 1 we have plotted the value of $| R_1 |$ versus the variable z. The values of $z = x$ where $| R_1 | = 0$ are easily seen to be multiples of π.

The second example

$$y^{iv} + 4y = -\lambda y'', \quad (4.4)$$

$$y(0) = y''(0) = 0 = y(x) = y''(x), \quad (4.5)$$

has the two eigenfunctions $\sin x$ and $\sin 2x$ when $\lambda = 5$. In this case the determinant of $| R_1 |$ is

$$| R_1 | = (1/2) \tan x \tan 2x = \sin^2 x / \cos 2x. \quad (4.6)$$

The above function has a double root at multiples of π indicating the existence of two linearly independent eigenfunctions. These results are illustrated in Fig. 2.

The last example to be considered is given by

$$-y^{vi} - 49y'' = \lambda(14y^{iv} + 36y), \quad (4.7)$$

$$y(0) = y''(0) = y^{iv}(0) = 0 = y(x) = y''(x) = y^{iv}(x). \quad (4.8)$$

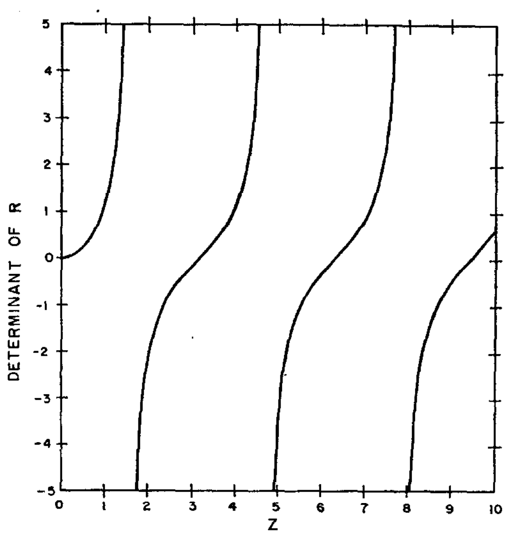

FIG. 1. Determinant of R_1 for simple eigenfunction case.

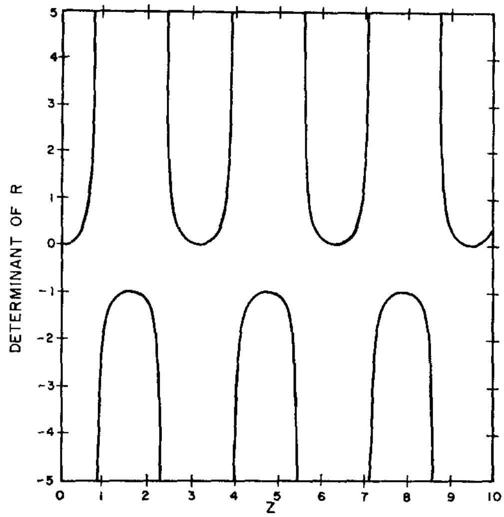

FIG. 2. Determinant of R_1 for double eigenfunction case.

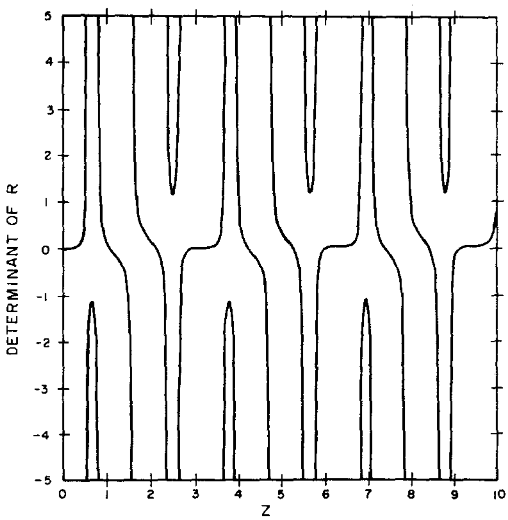

FIG. 3. Determinant of R_1 for triple eigenfunction case.

For $\lambda = 1$, this problem has the three eigenfunctions $\sin kx$, $k = 1, 2, 3$. Here the determinant of R_1 is given by

$$| R_1 | = (1/6) \tan x \tan 2x \tan 3x = \sin^3 x(3 - 4\sin^2 x)/3 \cos x \cos 2x(4\cos^2 x - 3). \quad (4.9)$$

The determinant of R_1 is plotted in Fig. 3. Notice that at multiples of π, the determinant of R_1 has a triple root indicating the existence of three linearly independent eigenfunctions for $\lambda = 1$.

5. Numerical Examples

We shall consider several examples of systems of various orders. Extreme accuracy was not attempted since the studies were primarily of a feasibility nature. All of the examples were generated with a single program utilizing a standard order Runge–Kutta integration scheme with fixed step size on a CDC-6600.

Example 1. We wish to discuss the calculation of the characteristic lengths for the problem

$$y^{(100)} = ky, \quad (5.1a)$$

$$y(0) = 0, \qquad y(x) = 0, \quad (5.1b)$$

$$y^{(2i)}(0) = 0, \qquad y^{(2i)}(x) = 0, \qquad i = 1, 2, \ldots, 50.$$

This example was chosen to demonstrate the feasibility of the method for high order systems which occur in the study of multigroup diffusion equations of neutron physics. For $k = 1$, the characteristic lengths are simply $x_n = n\pi$. The results for the first three characteristic lengths are

true value	calculated value
$x_1 = 3.141593$	3.141591
$x_2 = 6.283185$	6.283177
$x_3 = 9.424780$	9.424772

Example 2. We shall now demonstrate how the method can be used to compute the eigenvalues or characteristic lengths for certain integrodifferential equations. Consider the equation

$$(\operatorname{sgn} s) \frac{\partial n}{\partial z}(z, s) + a(s)\, n(z, s) = \lambda k(s) \int_{-1}^{1} n(z, s')\, ds', \qquad 0 \leqslant z \leqslant x, \quad (5.2a)$$

subject to the conditions

$$n(0, s) = 0, \qquad 0 < s < 1, \tag{5.2b}$$

$$n(x, s) = 0, \quad -1 < s < 0, \tag{5.2c}$$

where $a(s)$ and $k(s)$ are real piecewise continuous functions on $|s| \leqslant 1$. For fixed λ, we wish to compute the interval lengths x such that (5.2) has a nontrivial solution. Equations of the above form arise in the study of particle transport in a slab [5, 9, 11].

In order to apply our techniques to this type of problem we make the following substitutions

$$u(z, s) = n(z, s), \qquad \text{when } s > 0, \tag{5.3a}$$

$$v(z, s) = n(z, s), \qquad \text{when } s < 0. \tag{5.3b}$$

Then (5.2) can be written as

$$\frac{\partial u}{\partial z}(z, s) + a(s)\, u(z, s) = \lambda k(s) \left\{ \int_0^1 u(z, s')\, ds' + \int_{-1}^0 v(z, s')\, ds' \right\}, \tag{5.4a}$$

$$-\frac{\partial v}{\partial z}(z, s) + a(s)\, v(z, s) = \lambda k(s) \left\{ \int_0^1 u(z, s')\, ds' + \int_{-1}^0 v(z, s')\, ds' \right\}, \tag{5.4b}$$

$$u(0, s) = 0, \tag{5.4c}$$

$$v(x, s) = 0. \tag{5.4d}$$

The integral are replaced with some type of numerical quadrature, such as Gaussian quadrature, and then (5.4) becomes a system of ordinary differential equations of the form

$$\bar{u}' = A\bar{u} + B\bar{v}, \tag{5.5a}$$

$$-\bar{v}' = C\bar{u} + D\bar{v}, \tag{5.5b}$$

$$\bar{u}(0) = 0, \tag{5.5c}$$

$$\bar{v}(x) = 0. \tag{5.5d}$$

We shall consider two examples of this class of problem. The first example has been studied by Wing [16] and Allen [1]. Let

$$a(s) = |s|,$$

$$k(s) = e^{-5|s|}.$$

The results of our computation are compared in Table I with those obtained by Allen and Wing.

TABLE I

	Scott	Wing	Allen
λ	x_1	x_1	x_1
40	.12687	.13672	.1318
30	.16962	NR	.1766
20	.25581	.26562	.2680
10	.52000	.52930	.5452
5	1.0744	1.08398	1.1292
2	2.9617	2.97266	NR

The second example of the integrodifferential equation type is the Boltzmann equation for particle transport in a slab where

$$a(s) = 1/|s|,$$
$$k(s) = 1/2\,|s|.$$

We shall compare our results with two separate techniques. Wing [15] has performed extensive calculations for upper and lower bounds on the eigenvalues of this equation. We present two of his calculations

x	$\underline{\mu}_1$	$\bar{\mu}_1$
.2	.261	.262
2.0	.783	.785

where $\underline{\mu}_1$ and $\bar{\mu}_1$ are, respectively, lower and upper bounds on the first eigenvalue. We give our calculations below

$$\mu = .2610 \qquad x_1 = 0.199937$$
$$\mu = .2615 \qquad x_1 = 0.200533$$
$$\mu = .2620 \qquad x_1 = 0.201131,$$

and

$$\mu = .783 \qquad x_1 = 1.99978$$
$$\mu = .784 \qquad x_1 = 2.00927$$
$$\mu = .785 \qquad x_1 = 2.01883,$$

where $\mu = 1/\lambda$ in (5.20).

Our calculations are obviously consistent with those of Wing. The last comparison is made with the results of Carlson and Bell [5] on the calculation of the critical dimension of a particle multiplying medium in slab geometry. We emphasize that criticality type of computations are ideally suited to the invariant imbedding approach.

λ	Scott	Carlson and Bell
2.0	.62205	.6216

The result of Carlson and Bell was obtained using a very sophisticated variational approach.

6. Conclusions and Future Research

In this paper we have presented a mathematical technique for analyzing the eigenvalue or eigenlength problem for ordinary differential equations. The technique is capable of handling problems where the eigenvalue parameter appears in a nonlinear fashion, various types of singularities, and very general boundary conditions including periodic conditions [14]. In addition, the method generalizes to systems in a straightforward manner. All of the differential equations of the invariant imbedding technique are initial valued and tend to be quite stable numerically.

There are a number of features of the technique which warrant future study. An interesting generalization would be to problems having complex coefficients. These arise quite frequently in the study of stability of fluid flow.

In the case of second order problems, the boundary conditions must be periodic for an eigenvalue to have more than one eigenfunction associated with it. However, the periodicity is not necessary in the case of systems. There is very little known about when to expect degenerate eigenvalues for systems. It is possible that the analysis might be paralleled to that of linear algebraic equations using the formulation given by (3.5) or (3.12). The numerical experiments presented in Section 5 indicate that multiple eigenfunctions cause no particular problem.

Perhaps the most significant area for future research would be to the generalization of the ideas of this paper to the study of eigenvalues for partial differential equations.

References

1. R. C. Allen, Jr., *J. Math. Anal. Appl.* **30** (1970), 48–78.
2. D. W. Alspaugh, H. H. Kagiwada, and R. E. Kalaba, "Application of invariant imbedding to the eigenvalue problems for buckling of columns," The RAND Corp. RM-5954-PR, 1969.
3. F. V. Atkinson, "Discrete and Continuous Boundary Problems," Academic Press, New York, 1964.

4. J. M. Calvert and W. D. Davison, *J. Phys. Ser. A* **2** (1969), 278–292.
5. B. G. Carlson and G. I. Bell, *Proc. Intern. Conf. Peaceful Uses of Atomic Energy, Geneva*, **16** (1958), 535–549.
6. L. Collatz, "Eigenwertaufgaben mit Technischen Anwendungen," Akademische Verlagsgellschaft, Leipzig, 1963.
7. L. Fox, Some numerical experiments with eigenvalue problems in ordinary differential equations, in "Boundary Problems in Differential Equations" (R. E. Langer, ed.) Univ. Wisconsin Press, Madison, Wisconsin, 1960.
8. E. L. Ince, "Ordinary Differential Equations," Dover, New York, 1944.
9. P. Nelson, Jr. and M. R. Scott, *J. Math. Anal. Appl.* **34** (1971), 628–643.
10. M. R. Scott, L. F. Shampine, and G. M. Wing, *Computing* **4** (1969), 10–23.
11. M. R. Scott, *J. Math. Anal. Appl.* **28** (1969), 112–119.
12. M. R. Scott, "The application of invariant imbedding to characteristic value problems," Proc. Fourth. Hawaii Inter. Conf. System Sciences, Honolulu, Hawaii, Western Periodicals, 1971.
13. M. R. Scott, "A bibliography on invariant imbedding and related topics," Sandia Laboratories SC-B-71-0886, Albuquerque, New Mexico, 1971.
14. M. R. Scott, "An initial value method for the eigenvalue problem for ordinary differential equations," Sandia Laboratories, SC-RR-71-0791, Albuguerque, New Mexico, 1972.
15. G. M. Wing, *J. Math. Anal. Appl.* **11** (1965), 160–175.
16. G. M. Wing, "Mathematical Methods Suggested by Transport Theory" (R. E. Bellman and G. Birkhoff, eds.) Vol. 1, Proc. Symposia on Applied Mathematics, Amer. Math. Soc., 1969, pp. 159–177.